Decision Problems in Algebra

(FCUL Summer School)

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Outline

Fundamental Dehn's Decision Problems

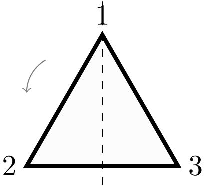
- Groups; presentations;
- The word problem;
- Dehn's Algorithm.

Undecidability

Related topics

Groups (The Dihedral Group)

D_n - the group of symmetries of a regular polygon of n sides;



$$\boxed{\text{Group: } \begin{array}{l} (xy)z = x(yz) \\ 1\,x = x \\ x^{-1}\,x = 1 \end{array}}$$

Rotations: $a =$(1 2 3), $a^2 =$(1 3 2), and id;
Reflections: $b =$(2 3), $ba =$(1 2), $ba^2 =$ (1 3).

- D_3 is generated by $\{a, b\}$; we write $D_3 = \langle a, b \rangle$;
- Relations in D_3: $a^3 = 1$, $b^2 = 1$, $(ba)^2 = 1$; any other relation in D_3 is a consequence of these: for example

$$\text{Is } ab \stackrel{?}{=} ba^2.\ \text{YES} : ab = b^2aba^3 = b(baba)a^2 = ba^2.$$

- D_3 is presented by $\langle a, b \mid a^3 = 1,\ b^2 = 1,\ (ba)^2 = 1 \rangle$.

Group presentations:

The Dihedral group D_3 can be given by the presentations:

$$\langle a, b \mid a^3 = 1,\ b^2 = 1,\ (ba)^2 = 1 \rangle$$

$$\langle a, b \mid a^3 = 1,\ b^2 = 1,\ ab = ba^2 \rangle$$

in the alphabet $\{a, b\}$. (Finite sequences of letters from the alphabet (for example, $aababaabb$) are called words).

If we consider symbols a^{-1} and b^{-1}, D_3 can also be given by:

$$\langle a, a^{-1}, b, b^{-1} \mid a^3 = 1, b^2 = 1, (ba)^2 = 1, aa^{-1} = a^{-1}a = 1, bb^{-1} = b^{-1}b = 1\rangle$$

In this case we can simply write:

$$\langle a, b \mid a^3,\ b^2,\ (ba)^2 \rangle_{gr}$$

assuming the relations $aa^{-1} = a^{-1}a = 1$ and $bb^{-1} = b^{-1}b = 1$.

Finitely presented groups

Example (The free group on 2 generators)

$$F_2 = \langle a, b \mid \ \rangle_{gr}.$$

Example (The Symmetric group on three letters)

$$S_3 = D_3 = \langle a, b \mid a^3,\ b^2,\ (ba)^2 \rangle.$$

Example (The free abelian group on 2 generators)

$$\mathbb{Z}^2 = \langle a, b \mid \ ab = ba \rangle_{gr} = \langle a, b \mid \ aba^{-1}b^{-1} \rangle.$$

Think of a as $(1, 0)$ and b as $(0, 1)$.

Definition

A group G is finitely presented (f.p.) if it is given by a finite presentation

$$\langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle.$$

The word problem

Given a f.p. group G, we have

Word problem for G

Find an algorithm with

input: a word w in the generators of G

output: YES or NO, according to whether w represents the identity in G.

More difficult:

The uniform word problem

Find an algorithm with

input: a f.p. group G, and a word w in the generators of G

output: YES or NO, according to whether w represents the identity in G.

Independence

The word problem is independent of the given f.p.

G has solvable word problem with respect to $\langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$

iff

G has solvable word problem with respect to $\langle x_1, \ldots, x_k \mid s_1, \ldots, s_p \rangle$.

Proof: Tietze transformations

Consider the group K given by the presentation $\langle a, b \mid abab^{-1} = 1 \rangle$.
We can transform the presentation by adding generators

$$\langle a, b, x, y \mid abab^{-1} = 1,\ x = ab,\ y = b^{-1} \rangle.$$

Introducing/deleting relations which are consequence of the others:

$$\langle a, b, x, y \mid x^2y^2 = 1,\ x = ab,\ y = b^{-1},\ a = xy \rangle.$$

Also delete generators: $\langle x, y \mid x^2y^2 = 1 \rangle_{gr}$.

Word problem for the free group F_n: $\langle a_1, \ldots, a_n \mid \rangle_{gr}$.

The word problem for F_n is decidable

For a word w in the generators:

1. Compute the reduced word $\overline{w}$;
2. Check if $\overline{w}$ is the empty word.

Example (The free group on 2 generators)

Let $w = abab^{-1}a^{-1}aba^{-1}b$.

Iteratively delete factors of the form aa^{-1}, $a^{-1}a$, bb^{-1} and $b^{-1}b$. We get

$$w = abab^{-1}\underline{a^{-1}a}ba^{-1}b \to aba\underline{b^{-1}b}a^{-1}b \to ab\underline{aa^{-1}}b \to abb,$$

and so $\overline{w} = abb$ is reduced.

Since $abb \neq 1$ we conclude that w does not represent the identity.

Word problem for D_3

Consider the presentation of D_3: $\langle a, b \mid a^3 \to 1,\ b^2 \to 1,\ ab \to ba^2 \rangle$.

The word problem for D_3 is decidable

For a word w in the generators $\{a, b\}$:

1. Compute the reduced word $\overline{w}$ such that $w \xrightarrow{*} \overline{w}$;
2. Check if $\overline{w}$ is the empty word.

Example (The free group on 2 generators)

Let $w = a^2b^2a$.
Compute $w = a^2\underline{b^2}a \to a^3 \to 1$, or
$w = a\underline{ab}ba \to \underline{ab}a^2ba \to ba^4ba \to b\underline{ab}a \to \underline{b^2}a^3 \to a^3 \to 1$,
so $\overline{w} = 1$, and we conclude that w represents the identity.

Why does it work? See last session.

Fundamental Dehn's decision problems

In 1911 Dehn proposed the following decision problems for f.p. groups:

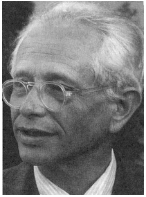
The Word Problem

The Conjugacy Problem

Find an algorithm with

input: words x, y in the generators of G

output: YES or NO, according to whether x and y are conjugate in G (i.e., exists $g \in G$ s.t. $x = gyg^{-1}$).



Max Dehn

The Isomorphism Problem

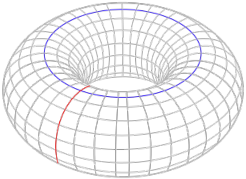
Find an algorithm with

input: f.p. $\langle X \mid R \rangle$ of G, and $\langle Y \mid S \rangle$ of H;

output: YES or NO, according to whether G and H are isomorphic.

Dehn's motivation: surface groups

The fundamental group $\pi_1(T)$ of a topological space T is an invariant.



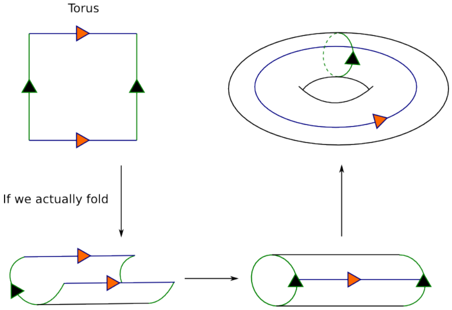
$\pi_1(T) = \{[\gamma] : \gamma \text{ is a loop with base point } x_0\}$;
$[\gamma]$ - classes of homotopic paths;
Paths are homotopic if one can be deformed to the other.

word problem $\Leftrightarrow$ whether or not a loop in T is contractible

conjugacy problem $\Leftrightarrow$ whether or not two loops are homotopic

The fundamental group of the Torus

Polygonal representation of the Torus:



The fundamental group of the torus $\pi_1(\mathbb{T}^2)$ is presented by $\langle a, b \mid [a, b] \rangle$, where $[a, b] = aba^{-1}b^{-1}$.

$\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$ has solvable word problem

Let $w = baba^{-1}baab^{-3}a^{-2}$. Since $[a, b] = 1 \Leftrightarrow ab = ba$, then $w = (aa^{-1}aaa^{-2})(bbbb^{-3}) = 1$.

Surface groups

An orientable compact surface of genus $g \geq 1$ has fundamental group given by a presentation:

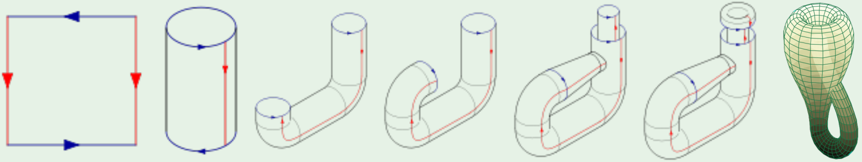
$$\langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle.$$

A nonorientable compact surface of genus $g \geq 1$ has fundamental group given by a presentation:

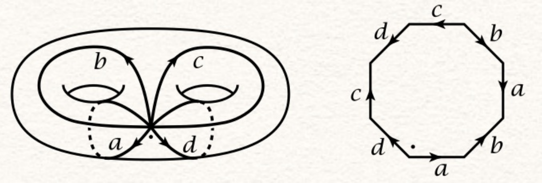
$$\langle a_1, \ldots, a_g \mid a_1^2 \ldots a_g^2 \rangle.$$

Example (The fundamental group of the Klein bottle)

We have seen that $\langle a, b \mid abab^{-1} = 1 \rangle$ and $\langle x, y \mid x^2y^2 = 1 \rangle_{gr}$ define the same group.



Dehn's algorithm



Theorem (Dehn 1912)

For any surface group of genus $g \geq 2$ there exists a Dehn Algorithm.

A group G on the generators A has a Dehn algorithm if:

- there exists a finite list of pairs $(u_1, v_1), \ldots, (u_n, v_n)$ with $|u_i| > |v_i|$; and
- if w is a reduced word representing the identity, then w contains some u_i as a factor.

Dehn's algorithm

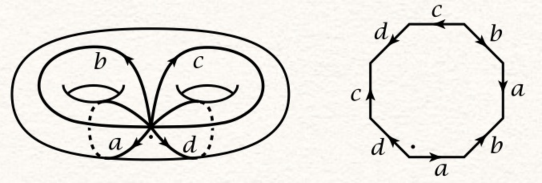
Dehn's algorithm to solve the word problem for G

Data: a word w in the generators of G
Result: YES (if $w = 1$) or NO (if $w \neq 1$)

```
while w is not the trivial word do
    freely reduce w;
    if some u_i is a factor of w then
        replace that instance by v_i;
    else
        return NO;
    end
end
return YES;
```

Dehn's algorithm: example

Consider the double torus:



with fundamental group D given by $\langle a, b, c, d \mid [a, b][c, d] \rangle$.
The set Δ of all "cyclic permutations" of $[a, b][c, d]$ is a Dehn Algorithm for D.
For example

$$w = a^2 b^{-1} (a^{-1} d c d^{-1} c^{-1})^2 b \to a^2 b^{-1} (b a^{-1} b^{-1})^2 b$$

and $\overline{a^2 b^{-1} (b a^{-1} b^{-1})^2 b} = 1$.